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On LA-Semimodule Over LA-Semiring

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Abstract

In this paper, we develop an LA-module over LA-ring to a new concept namely LA-semimodule over LA-semiring. Let S be a non-empty set with two binary operations "$+$" and "$*$". Set S is called a left almost semiring (LA-semiring) if $(S, +)$ is an LA-semigroup, $(S, *)$ is an LA-semigroup and satisfying left and right distributive law of "$*$" over "$+$" hold. Let $(S, +, *)$ is an LA-semiring with left additive identity equal to 0_S and left multiplicative identity equal to 1, non-empty set M is called an LA-semimodule over S if 1) $(M, +)$ is an LA-semigroup with left identity, 2) the map $S \times M \rightarrow M$, $(s, m) \mapsto sm$ where $s \in S$ and $m \in M$ satisfies i) $s(m+n) = sm + sn$, ii) $(r + s)m = rm + sm$, iii) $r(sm) = s(rm)$, iv) $1 * m = m$, for all $r, s \in R$, and $m, n \in M$. Then, we investigate the basic properties and the Isomorphims Theorem for LA-semimodule over LA-semiring.

Keywords: LA-semigroup; LA-semiring; LA-semimodule.

1 Introduction

The concept of AG-groupoid is a generalization of commutative semigroup concept without associative law that introduced by [4]. A grupoid S is called AG-groupoid if its element satisfy the left invertive law i.e $(ab)c = (cb)a$ for all $a, b, c \in S$. In [1], AG-groupoid is also known as left almost semigroup (LA-semigroup). A groupoid G is called medial if G satisfy the medial law i.e $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in G$. A groupoid G is called paramedial if satisfy the paramedial law i.e $(ab)(cd) = (db)(ca)$ for all $a, b, c, d \in G$ [1]. An LA-semigroup S always satisfies the medial law [[1], Lemma 1.1(i)] while an LA-semigroup S with left identity e always satisfies the paramedial law [[1], Lemma 1.2 (ii)]. An LA-semigroup S with left identity e also satisfies $a(bc) = b(ac)$ for all $a, b, c \in S$ [[6], Lemma 4].

The works of [3] and [5] extend the notion of LA-semigroup into LA-group. An LA-semigroup G is called an LA-group if there exists left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for all $a \in G$ there exists $a^{-1} \in G$ such that $a^{-1}a = aa^{-1} = e$. Then, [8] give the properties of cancellative LA-semigroup. An element a of an LA-semigroup S is called left cancellative if $ax = ay$ implies $x = y$ for all $x, y \in S$. Similarly, an element a of an LA-semigroup S is called right cancellative if $xa = ya$ imples $x = y$ for all $x, y \in S$. An element a of an LA-semigroup S is called cancellative if it is both left and right cancellative. An LA-semigroup S is called left cancellative if every element of S is left cancellative. Similarly, an LA-semigroup S is called right cancellative if every element of S is right cancellative and it is called cancellative if every element of S is cancellative. A finite cancellative LA-semigroup is an LA-group [8].

In 2011, [9] extended LA-group to a non-associative structure with respect to both binary operations $'+'$ and $'\cdot'$ namely left almost ring (LA-ring). A left almost ring means a nonempty set R with at least two element such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup and both left and right distributive laws hold.

Next, [10] extended LA-group and LA-ring concept to LA-module. Let $(R, +, \cdot)$ be an LA-ring with left identity 1. An LA-group $(M, +)$ is called an LA-module over R, if the map $R \times M \to M$ is defined $(r, m) \mapsto rm \in M$, and where $r \in R$ and $m \in M$ satisfies : $r(m_1 + m_2) = rm_1 + rm_2$, $(r_1 + r_2)m = r_1m + r_2m$, $r_1(r_2m) = r_2(r_1m)$, $1 \cdot m = m$, for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$.

Semiring S is a non-empty set with two binary operation that satisfy $(S, +)$ is monoid commutative, $(S, \cdot)$ is semigroup, both left and right distributive laws hold [2]. Then, [2] give the definition of semimodule over semiring and some property of it. After that, [7] extend LA-ring and semiring into LA-semiring. In this paper, we will generalize LA-module over LA-ring into LA-semimodule over LA-semiring. Then, we investigate the properties of LA-semimodule over LA-semiring, along with all that associated with LA-semimodule.

2 Result and Discussion

2.1 LA-Semimodule

In this section, we give the definition of an LA-semimodule over an LA-semiring. we study some examples of LA-semimodule and discuss the elementary properties of an LA-semimodule.

Definition 2.1 ([7]). *A left almost semiring (LA-semiring) is a non empty set S with two binary operations " $+$ " and " $*$ " that satisfying the following conditions:*

i. $(S,+)$ *is an LA-semigroup.*

ii. $(S,*)$ *is an LA-semigroup.*

iii. *Both left and right distributive laws holds:*
$x*(y+z) = x*y + x*z$
$(y+z)*x = y*x + z*x$
for all $x,y,z \in S$.

In this paper, all LA-semiring S are LA-semiring with left additive identity equal to 0_S and left multiplicative identity equal to 1.

Example 2.1. *Here some examples of LA-semiring:*

i. *All LA-ring are LA-semiring.*

ii. *Let* $S = \mathbb{Z}_n$, $n \in \mathbb{N}$ *and define binary operation*

$$\begin{aligned} \ominus : S \times S &\to S \\ (a,b) &\mapsto a \ominus b = b - a, \end{aligned}$$

and

$$\begin{aligned} * : S \times S &\to S \\ (a,b) &\mapsto a * b = ab. \end{aligned}$$

Note that for any $a,b,c \in S$, *we have*

$$\begin{aligned} (a \ominus b) \ominus c &= c - b + a \\ &= a - b + c \\ &= (c \ominus b) \ominus a. \end{aligned}$$

So, $(S,\ominus)$ *is an LA-semigroup. Furthermore, note that*

$$(a \ominus b) \ominus c = c - b + a \neq c - b - a = a \ominus (b \ominus c)$$

and

$$a \ominus b = b - a \neq a - b = b \ominus a.$$

Hence, $(S,\ominus)$ *is not a commutative semigroup. Since* $(S,*)$ *is a commutative monoid then* $(S,*)$ *is an LA-semigroup. Next, note that*

$$\begin{aligned} (a \ominus b)c = (b-a)c = bc - ac = ac \ominus bc \\ a(b \ominus c) = a(c-b) = ac - ab = ab \ominus ac \end{aligned}$$

for any $a,b,c \in S$. *Therefore,* $(S,\ominus,*)$ *is an LA-semiring.*

Definition 2.2. *Let $(S, +, *)$ be an LA-semiring. A set M is called LA-semimodule over LA-semiring S if satisfies:*

i. *$(M, +)$ is an LA-semigroup with left identity.*

ii. *Defined map $\cdot : S \times M \to M$ where $(r, m) \mapsto rm$, $r \in S$, $m \in M$ and satisfies:*

  (a) $r(m + n) = rm + rn$

  (b) $(r + s)m = rm + sm$

  (c) $r(sm) = s(rm)$

  (d) $1 \cdot m = m$, *for all $r, s \in S$ and $m, n \in M$.*

In this paper, all LA-semimodule M are LA-semimodule with left identity equal to 0_M.

Example 2.2. *Here some examples of LA-semimodule:*

i. *All LA-module over LA-ring R are LA-semimodule over R.*

ii. *All LA-semiring S are LA-semimodule over itself.*

Theorem 2.1. *Let $(M, +)$ be a cancellative LA-semimodule over LA-semiring $(S, +, *)$. Then, for all $s \in S$ and $a \in M$ satisfies:*

i. $s \cdot 0_M = 0_M$

ii. $0_S \cdot a = 0_M$

Proof. Let a be an arbitrary element in M and s be an arbitrary element in S, then the following conditions are hold:

i. Since M is an LA-semimodule with left identity 0_M then

$$\begin{aligned} s \cdot 0_M = s(0_M + 0_M) &\Leftrightarrow s \cdot 0_M = s \cdot 0_M + s \cdot 0_M \\ &\Leftrightarrow 0_M + s \cdot 0_M = s \cdot 0_M + s \cdot 0_M \end{aligned}$$

since M is cancellative then $0_M = s \cdot 0_M$.

ii. Since S is an LA-semiring with left additive identity 0_S then

$$\begin{aligned} 0_S \cdot a = (0_S + 0_S)a &\Leftrightarrow 0_S \cdot a = 0_S \cdot a + 0_S \cdot a \\ &\Leftrightarrow 0_M + 0_S \cdot a = 0_S \cdot a + 0_S \cdot a \end{aligned}$$

since M is cancellative then $0_M = 0_S \cdot a$.

□

2.2 LA-Subsemimodule

In this section, we give the definition of an LA-subsemimodule of LA-semimodule. Then, we initiate the following definition.

Definition 2.3. *Let M be an LA-semimodule over LA-semiring S and N be a non empty subset of M. LA-subsemigroup N is called LA-subsemimodule over S, if $SN \subseteq N$, i.e , $sn \in N$, for all $s \in S$ and $n \in N$.*

Remark 2.1. *Let M be an LA-semimodule over LA-semiring S. Then M it self and $\{0\}$ are LA-subsemimodule over LA-semiring S and its called improper LA-subsemimodule.*

Corollary 2.1. *Let M be a cancellative LA-semimodule over LA-semiring S and N be an LA-subsemimodule of M. Then, N cancellative and $0_M \in N$.*

Proof. The first statement is clear. The second statement, let a be an arbitrary element in M. Since N is an LA-subsemimodule of M and M is a cancellative LA-semimodule then $0_M = 0_S \cdot a \in N$. □

Theorem 2.2. *Let M be a cancellative LA-semimodule over LA-semiring S. If N_i are LA-subsemimodule of M for $i = 1, 2, 3, ..., n$, then $\bigcap_{i=1}^{n} N_i$ is an LA-subsemimodule of M.*

Proof. Since N_i is an LA-subsemimodule then $0_M \in N_i$ for all $i = 1, 2, 3, \ldots, n$. Hence, $\bigcap_{i=1}^{n} N_i \neq \emptyset$. Clear that $\bigcap_{i=1}^{n} N_i \subseteq M$. Let $a, b \in \bigcap_{i=1}^{n} N_i$, then $a, b \in N_i$ for all $i = 1, 2, 3, \ldots, n$. Since N_i are an LA-subsemimodule, then $a + b \in N_i$ and $sa \in N_i$ for all $s \in S$, $i = 1, 2, \ldots, n$. As a consequence $sa \in \bigcap_{i=1}^{n} N_i$. Hence $\bigcap_{i=1}^{n} N_i$ also an LA-subsemimodule of M. □

Theorem 2.3. *Let M be an LA-semimodule over LA-semiring S. If N_i are a subsemimodule of M for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} N_i$ is a subsemimodule.*

Proof. Since N_i is an LA-subsemimodule then $N_i \neq \emptyset$ for all $i = 1, 2, 3, \ldots, n$. As a consequence $\sum_{i=1}^{n} N_i \neq \emptyset$ and $\sum_{i=1}^{n} N_i \subseteq M$. Let $a, b \in \sum_{i=1}^{n} N_i$ where $a = a_1 + a_2 + ... + a_n$ and $b = b_1 + b_2 + ... + b_n$ with $a_i, b_i \in N_i$ for all $i = 1, 2, 3, \ldots, n$. Since N_i are LA-subsemimodule, then we have

$$
\begin{aligned}
a + b &= (a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n) \\
&= ((a_1 + ... + a_{n-1}) + a_n) + ((b_1 + ... + b_{n-1}) + b_n) \\
&= ((a_1 + ... + a_{n-1}) + (b_1 + ... + b_{n-1})) + (a_n + b_n) \\
&= ((a_1 + ... + a_{n-2} + a_{n-1}) + (b_1 + ... + b_{n-2}) + b_{n-1}) + (a_n + b_n) \\
&= ((a_1 + ... + a_{n-2} + (b_1 + ... + b_{n-2})) + (a_{n-1} + b_{n-1}) + (a_n + b_n) \\
&= (((a_1 + b_1) + (a_2 + b_2)) + ...) + (a_n + b_n) \\
&= (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n).
\end{aligned}
$$

Since $a_i + b_i \in N_i$ then $a + b \in \sum_{i=1}^{n} N_i$. Hence, $\sum_{i=1}^{n} N_i$ is an LA-subsemigroup. Next, let s be an arbitrary element in S, then

$$sa = s(a_1 + a_2 + ... + a_n) = sa_1 + sa_2 + \ldots + sa_n \in \sum_{i=1}^{n} N_i.$$

Hence $\sum_{i=1}^{n} N_i$ also LA-subsemimodule of M. □

Definition 2.4. *Let M be an LA-semimodule over LA-semiring S and N is an LA-subsemimodule of M. $M/N = \{a + N : a \in M\}$ is called a quotient LA-semimodule.*

Note that the binary operation in quotient LA-semimodule M/N are $'+'$ and $'\cdot'$. Since M is medial then we have

$$\begin{aligned}(a + N) + (b + N) &= (a + b) + (N + N)\\ &= (a + b) + N.\end{aligned}$$

Since M is an LA-semimodule over S, and N is an LA-subsemimodule then

$$s(a + N) = sa + sN = sa + N.$$

Since M contains left identity and M satisfy medial law then

$$N + (a + N) = (0 + N) + (a + N) = (0 + a) + (N + N) = a + N.$$

For any $a + N, b + N \in M/N$ and $s \in S$. Hence, N is left identity element in M/N.

Proposition 2.1. *Let M be an LA-semimodule over LA-semiring S and N be an LA-subsemimodule of M. If M is cancellative then M/N is cancellative.*

Proof. Let $a + N, b + N$ and $c + N$ be arbitrary elements in M/N then

$$\begin{aligned}(a + N) + (c + N) = (b + N) + (c + N) &\Rightarrow (N + N) + (c + a) = (N + N) + (c + b)\\ &\Rightarrow N + (c + a) = N + (c + b)\\ &\Rightarrow (0 + N) + (c + a) = (0 + N) + (c + b)\\ &\Rightarrow (0 + c) + (N + a) = (0 + c) + (N + b)\\ &\Rightarrow c + (N + a) = c + (N + b)\\ &\Rightarrow N + a = N + b\\ &\Rightarrow (0 + N) + a = (0 + N) + b\\ &\Rightarrow (a + N) + 0 = (b + N) + 0\\ &\Rightarrow (a + N) = (b + N).\end{aligned}$$

Thus, M/N is right cancellative. Since M/N is an LA-semigroup with left identity then M/N is left cancellative too. Therefore, M/N is cancellative. □

2.3 LA-Semimodule Homomorpishm

In this section, we give the definition of LA-semimodule homomorpishm and its basic properties.

Definition 2.5. *Let M, M' be LA-semimodule over LA-semiring S. A map $\varphi : M \to M'$ is called LA-semimodule homomorpishm if for any $s \in S$ and $m, n \in M$, satisfies the following conditions:*

i. $\varphi(m+n) = \varphi(m) + \varphi(n)$

ii. $\varphi(sm) = s\varphi(m)$

Corollary 2.2. *Let M, M' be LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorphism. If M and M' are cancellative then $\varphi(0_M) = 0_{M'}$.*

Proof. Let a be an arbitrary element in M and $x \in M'$ where $x = \varphi(a)$, then

$$\begin{aligned}\varphi(a) = x &\Leftrightarrow 0_S\varphi(a) = 0_S \cdot x\\ &\Leftrightarrow \varphi(0_S \cdot a) = 0_{M'}\\ &\Leftrightarrow \varphi(0_M) = 0_{M'}.\end{aligned}$$

□

Remark 2.2 ([8])**.** *LA-semigroup M is an LA-group iff M is finite cancellative.*

Lemma 2.1. *Let M, M' be finite cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorphism. If M and M' are finite cancellative then $\varphi(-a) = -\varphi(a)$ for all $a \in M$.*

Proof. Let a be an arbitrary element in M. Since M is finite cancellative then there exist $-a \in M$ such that $-a + a = 0_M$. Since φ is LA-semimodule homomorphism and M' is finite cancellative then

$$\begin{aligned}\varphi(-a+a) = \varphi(0_M) &\Leftrightarrow \varphi(-a) + \varphi(a) = 0_{M'}\\ &\Leftrightarrow \varphi(-a) + \varphi(a) - \varphi(a) = 0_{M'} - \varphi(a)\\ &\Leftrightarrow \varphi(-a) = -\varphi(a).\end{aligned}$$

□

Theorem 2.4. *Let M, M' be cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorpishm, then the following conditions are holds:*

i. If P is an LA-subsemimodule of M, then $\varphi(P)$ is an LA-subsemimodule of M'.

ii. If Q is an LA-subsemimodule of M', then $\varphi^{-1}(Q)$ is an LA-subsemimodule of M.

Proof. Note that

$$\begin{aligned}\varphi(P) &= \{x \in M' \mid x = \varphi(a),\ a \in P\}\\ \varphi^{-1}(Q) &= \{a \in M \mid \varphi(a) \in Q\}.\end{aligned}$$

Then, consider that

i. Since P is an LA-subsemimodule and φ is an LA-semimodule homomorphism then $P \neq \emptyset$ and $\varphi(P) \neq \emptyset$. Clear that $\varphi(P) \subseteq M'$. Let x, y be two arbitrary elements in $\varphi(P)$ where $x = \varphi(a), y = \varphi(b), a, b \in P$ then $x + y = \varphi(a) + \varphi(b) = \varphi(a + b)$. Since P is an LA-subsemimodule then $a + b \in P$. Hence, $x + y \in \varphi(P)$ and $\varphi(P)$ is an LA-subsemigroup. Let s be an arbitrary element in S, then $sx = s\varphi(a) = \varphi(sa)$. Since P is an LA-subsemimodule then $sa \in P$. So, $sx \in \varphi(P)$. Therefore, $\varphi(P)$ is an LA-subsemimodule of M'.

ii. Since Q is an LA-subsemimodule of M' and M' is a cancellative LA-semimodule then $0_{M'} \in Q$. Hence, $0_{M'} = \varphi(a)$ implies $a = 0_M$, then $\varphi^{-1}(Q) \neq \emptyset$. Clear that $\varphi^{-1}(Q) \subseteq M$. Let a, b be two arbitrary element in $\varphi^{-1}(Q)$, then $\varphi(a + b) = \varphi(a) + \varphi(b) \in Q$. Hence, $a + b \in \varphi^{-1}(Q)$ and $\varphi^{-1}(Q)$ is an LA-subsemigroup. Let $s \in S$ then $\varphi(sa) = s\varphi(a)$. Since $\varphi(a) \in Q$ and Q is an LA-subsemimodule then $s\varphi(a) \in Q$. Therefore, $\varphi^{-1}(Q)$ is an LA-subsemimodule of M.

□

Definition 2.6. *Let $\varphi : M \to M'$ be an LA-semimodule homomorpishm. Kernel of φ is defined by $Ker(\varphi) = \{m \in M : \varphi(m) = 0\}$ and image of φ is defined by $Im(\varphi) = \{\varphi(m) : m \in M\}$.*

Lemma 2.2. *Let M, M' be cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorpishm, then $Ker(\varphi)$ and $Im(\varphi)$ are LA-subsemimodule of M and M', respectively.*

Proof. Since $\varphi(0_M) = 0_{M'}$ then $Ker(\varphi) \neq \emptyset$. Clear that $Ker(\varphi) \subseteq M$. Let a, b be two arbitary element in $Ker(\varphi)$ then $\varphi(a + b) = \varphi(a) + \varphi(b) = 0_{M'}$. Hence, $a + b \in Ker(\varphi)$ and $Ker(\varphi)$ is an LA-subsemigroup of M. Let s be an arbitary element in S, then $\varphi(sa) = s\varphi(a) = s \cdot 0_{M'} = 0_{M'}$. Therefore, $sa \in Ker(\varphi)$ and $Ker(\varphi)$ is an LA-subsemimodule of M.

Next, since $\varphi(0_M) = 0_{M'}$, then $Im(\varphi) \neq \emptyset$. Clear that $Im(\varphi) \subseteq M'$. Let x, y be two arbitary element in $Im(\varphi)$ where $x = \varphi(a), y = \varphi(b)$, $a, b \in M$ then $x + y = \varphi(a) + \varphi(b) = \varphi(a + b)$. Since $a + b \in M$ then $x + y \in Im(\varphi)$ and $Im(\varphi)$ is an LA-subsemigroup of M'. Let s be an arbitrary element in S then $sx = s\varphi(a) = \varphi(sa)$. Since M is an LA-subsemimodule then $sa \in M$. Hence, $sx \in Im(\Phi)$ and $Im(\Phi)$ is an LA-subsemimodule of M'. □

Proposition 2.2. *Let M, M' be finite cancellative LA-semimodule over LA-semiring S and map $\varphi : M \to M'$ be an LA-semimodule homomorpishm. Map φ is one-one if and only if $Ker(\varphi) = \{0_M\}$.*

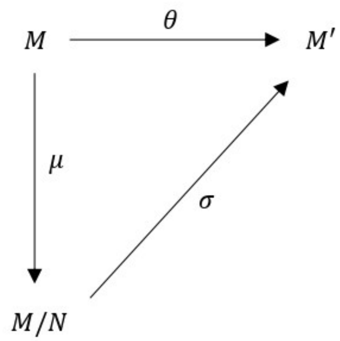
Proof. ($\Rightarrow$) Let a be an arbitrary element in $Ker(\varphi)$, then $\varphi(a) = 0_{M'}$. Since M, M' are cancellative LA-semimodule and φ is an LA-semimodule homomorphism then $\varphi(0_M) = 0_{M'}$. Since φ is one-one and $\varphi(a) = 0_{M'} = \varphi(0_M)$, then $a = 0_M$. Finally, $ker(\varphi) = \{0_M\}$.

($\Leftarrow$) Let $Ker(\varphi) = \{0_M\}$ and a, b be two arbitrary elements in M such that $\varphi(a) = \varphi(b)$. Since M, M' are finite cancellative LA-semigroup, then M, M' are LA-group. Hence, $\varphi(a) = \varphi(b)$ implies $\varphi(a) - \varphi(b) = 0_{M'}$. Since φ is an LA-semimodule homomorpishm, then $\varphi(a - b) = 0_{M'}$. Hence, $a - b \in Ker(\varphi)$. Since $Ker(\varphi) = \{0_M\}$ then $a - b = 0_M$. As consequence, $a = b$. So, φ is one-one. □

2.4 Isomorpishm Theorem for LA-Semimodule

In this section, we discuss about the isomorphism theorem in LA-semimodule over LA-semiring.

Theorem 2.5. *Let M, M' be finite cancellative LA-semimodule over LA-semiring S, $\theta : M \to M'$ be an LA-semimodule epimorphism, and $\mu : M \to M/Ker(\theta)$ be a natural LA-semimodule homomorphism. Then, there exist an LA-semimodule isomorphism $\sigma : M/N \to M'$, where $N = Ker(\theta)$ and make the diagram below commute.*



Proof. Since M, M' is a cancellative LA-semimodule and θ is an LA-semimodule homomorphism then $N = Ker(\theta)$ is an LA-subsemimodule of M. Therefore, M/N is a quotient LA-semimodule. Next, consider the mapping

$$\begin{aligned}\sigma : M/N &\to M'\\ a + N &\mapsto \sigma(a+N) = \theta(a) = a'.\end{aligned}$$

Then, we will show that σ is an isomorphism. Note that,

i. First, we will show that the mapping is well defined. Since M and M' are finite cancellative LA-semimodule then M/N is a finite cancellative quotient LA-semimodule. Hence, M/N and M' are LA-semigroup. Let $a + N, b + N$ be two arbitrary elements in M/N such that $a + N = b + N$, then

$$\begin{aligned}a + N = b + N &\Rightarrow a - b + N = N\\ &\Rightarrow a - b \in N\\ &\Rightarrow \theta(a-b) = 0\\ &\Rightarrow \theta(a) - \theta(b) = 0\\ &\Rightarrow \theta(a) = \theta(b)\\ &\Rightarrow \sigma(a+N) = \sigma(b+N).\end{aligned}$$

Thus σ is well defined.

ii. Let $a + N$ and $b + N$ be two arbitrary elements of M/N such that $\sigma(a+N) = \sigma(b+N)$, then

$$\begin{aligned}\sigma(a+N) = \sigma(b+N) &\Rightarrow \theta(a) = \theta(b)\\ &\Rightarrow \theta(a) - \theta(b) = 0\\ &\Rightarrow \theta(a-b) = 0\\ &\Rightarrow a - b \in N\\ &\Rightarrow a - b + N = N\\ &\Rightarrow a + N = b + N.\end{aligned}$$

Therefore, σ is one-one.

iii. Next we will show that σ is onto. Let a' be an arbitary element of M'. Since θ is an epimorphism from M to M', then there is an element a in M such that $\theta(a) = a'$. Since $\theta(a)$ being the σ-image of the coset $a + N$ in M/N, then $a' = \theta(a) = \sigma(a + N)$. Thus, σ is onto.

iv. Finally, σ is an LA-semimodule homomorpishm, i.e

(a) Let $a + N, b + N$ be two arbitrary elements in M/N then

$$\begin{aligned}\sigma[(a+N)+(b+N)] &= \sigma[(a+b)+N]\\ &= \theta(a+b)\\ &= \theta(a)+\theta(b)\\ &= \sigma(a+N)+\sigma(b+N).\end{aligned}$$

(b) Let $a + N$ be an arbitrary element in M/N and s be an arbitrary element in S then

$$\begin{aligned}\sigma[s(a+N)] &= \sigma(sa+N)\\ &= \theta(sa)\\ &= s\theta(a)\\ &= s\sigma(a+N).\end{aligned}$$

Hence, σ is an LA-semimodule isomorpishm from M/N to M' or $M/N \cong M'$. □

Theorem 2.6. *Let M be a finite cancellative LA-semimodule over LA-semiring S. If I and J are LA-subsemimodule of M, then $\frac{I+J}{J} \cong \frac{I}{I\cap J}$.*

Proof. Since I and J are LA-subsemimodule of M, then $I+J$ is an LA-subsemimodule of M. Since $J \subseteq I + J$ and J is an LA-subsemimodule, then $\frac{I+J}{J}$ is a quotient LA-semimodule. Since I and J are LA-subsemimodule, then $I \cap J$ is an LA-subsemimodule. Since $I \cap J \subseteq I$, then $\frac{I}{I\cap J}$ is a quotient LA-semimodule. Next, define a mapping

$$\begin{aligned}\theta : I &\to \frac{I+J}{J}\\ a &\mapsto \theta(a) = (a+0)+J = a+J.\end{aligned}$$

We will prove this theorem by using Theorem 2.5, then note that

i. Clear that φ is well defined. Let a, b be two arbitrary elements in I and s be an arbitrary element in S, then

(a) $\theta(a + b) = (a + b) + J = (a + J) + (b + J) = \theta(a) + \theta(b)$.

(b) $\theta(sa) = (sa) + J = s(a + J) = s\theta(a)$.

Thus, θ is an LA-semimodule homomorpishm. Next, note that for any $a + J \in \frac{I+J}{J}$, then exists $a \in I$ such that $\theta(a) = a + J$. Therefore, θ is an onto homomorpishm.

ii. Since J is a left identity in quotient LA-semimodule $\frac{I+J}{J}$, then

$$\begin{aligned}Ker(\theta) &= \{a \in I : \theta(a) = J\}\\ &= \{a \in I : a + J = J\}\\ &= \{a \in I : a \in J\}\\ &= \{a \in I \cap J\}\\ &= I \cap J.\end{aligned}$$

Since θ is an LA-semimodule epimorphism, $Ker(\theta) = I \cap J$ and M is finite cancellative then by Theorem 2.5 we have $\frac{I+J}{J} \cong \frac{I}{I \cap J}$. $\square$

Theorem 2.7. *Let M be a finite cancellative LA-semimodule over LA-semiring S. If J and K are LA-subsemimodule of M, where $J \subseteq K$, then $\frac{M/J}{K/J} \cong \frac{M}{K}$.*

Proof. Clear that M/J, M/K and K/J are quotient LA-semimodule over S. Since $K \subseteq M$ then $K/J \subseteq M/J$. Hence, K/J is an LA-subsemimodule of M/J and $\frac{M/J}{K/J}$ is an quotient LA-semimodule. Define a mapping

$$\begin{aligned} \theta : M/J &\to M/K \\ a + J &\mapsto \theta(a+J) = a + K. \end{aligned}$$

Then, note that

i. Since M is cancellative then M/J and M/K are cancellative. Hence, $\theta(J) = K$ implies θ is well defined. Then, we will show that θ is an LA-semimodule homomorphism

   (a) For any $a + J, b + J \in M/J$, we have

$$\begin{aligned} \theta[(a+J)+(b+J)] &= \theta[(a+b)+J] \\ &= (a+b)+K \\ &= (a+K)+(b+K) \\ &= \theta(a+J)+\theta(b+J). \end{aligned}$$

   (b) For any $a + J \in M/J$ and $s \in S$, we have

$$\begin{aligned} \theta[s(a+J)] &= \theta(sa+J) \\ &= sa+K \\ &= s(a+K) \\ &= s\theta(a+J). \end{aligned}$$

   Hence, θ is an LA-semimodule homomorpishm.

   Furthermore, since $J \subseteq K$ then for any $a + K \in M/K$, we can choose $a + J \in M/J$ such that $\theta(a + J) = a + K$. Thus, θ is an epimorphism.

ii. We will show that $Ker(\theta) = K/J$, then consider that

$$\begin{aligned} Ker(\theta) &= \{a+J \in M/J : \theta(a+J) = K\} \\ &= \{a+J \in M/J : a+K = K\} \\ &= \{a+J \in M/J : a \in K\} \\ &= \{a+J \in K/J\} = K/J. \end{aligned}$$

Now, since θ is an LA-semimodule epimorphism, $Ker(\theta) = K/J$ and $M/J, M/K$ are finite cancellative, then by Theorem 2.5 we have $\frac{M/J}{K/J} \cong \frac{M}{K}$. $\square$

3 Conclusions

Any LA-semimodule over LA-semiring are satisfy The First Isomorphism Theorem, The Second Isomorphism Theorem and The Third Isomorphism Theorem. Also, LA-semimodule over LA-semiring are satisfy some properties like properties of module over ring.

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References

[1] J. R. Cho, J. Jezek & T. Kepka (1999). Paramedial groupoids. *Czechoslovak Mathematical Journal*, *49*(2), 277–290.

[2] J. S. Golan (2013). *Semirings and their applications*. Springer Science & Business Media, Haifa, Israel.

[3] M. S. Kamran (1993). *Conditions for LA-Semigroups To Resemble Associative Structures*. PhD thesis, Quad-i-Azam University, Islamabad, Pakistan.

[4] M. A. Kazim & M. Naseerudin (1977). On almost semigroups. *Portugaliae Mathematica*, *36*, 41–47.

[5] Q. Mushtaq & M. S. Kamran (1996). On left almost groups. *Proceedings of the Pakistan Academy of Sciences*, *33*, 1–2.

[6] Q. Mushtaq & S. M. Yusuf (1979). On locally associative LA-semigroups. *Journal of Natural Sciences and Mathematics*, *XIX*(1), 57–62.

[7] K. Rahman, F. Husain, S. Abdullah & M. S. A. Khan (2016). Left almost semirings. *International Journal of Computer Science and Information Security*, *14*(9), 201–216.

[8] M. Shah, T. Shah & A. Ali (2011). On the cancellativity of AG-groupoids. *International Mathematical Forum*, *6*(44), 2187–2194.

[9] M. Shah & T. Shah (2011). Some basic properties of LA-ring. *International Mathematical Forum*, *6*(44), 2195–2199.

[10] T. Shah, M. Raees & G. Ali (2011). On LA-modules. *International Journal of Contemporary Mathematical Sciencess*, *6*(21), 999–1006.